1997

Reconstructing the Unity of Mathematics circa 1900

David J. Stump
University of San Francisco, stumpd@usfca.edu

Reconstructing the Unity of Mathematics circa 1900

David J. Stump
University of San Francisco

Standard histories of mathematics and of analytic philosophy contend that work on the foundations of mathematics was motivated by a crisis such as the discovery of paradoxes in set theory or the discovery of non-Euclidean geometries. Recent scholarship, however, casts doubt on the standard histories, opening the way for consideration of an alternative motive for the study of the foundations of mathematics—unification. Work on foundations has shown that diverse mathematical practices could be integrated into a single framework of axiomatic systems and that much of mathematics could be expressed in a single language. The new framework was the product of an interdisciplinary coalition whose ideas resemble those later adopted by the Vienna Circle and logical empiricists.

Without unity existence itself cannot be sustained. (BOETHIUS)

I. Introduction

The standard accounts of the history of the foundations of mathematics claim that the investigation of foundations was motivated by the discovery of paradoxes in set theory, a story of crisis that leads one to believe that the primary motivation for the study of the foundations of

I would like to thank Zdenek Hedrlín, Don Howard, Masato Mitsuda, Eric Palmer, George Reisch, Andrea Smith, and an anonymous referee for helpful comments on this paper and the University of San Francisco faculty development fund for support. Half of this article was developed at an National Endowment for the Humanities Summer Seminar directed by Rachel Laudan, and earlier versions of parts of the article were read at the History of Science Society meeting in 1994 and at the History of the Philosophy of Science meeting in 1996.

Perspectives on Science 1997, vol. 5, no. 3

mathematics was epistemological and originated in response to doubts about the consistency of mathematics and the truth of some of its branches. This type of presentation frequently is found in histories and textbooks (Curry 1963, pp. 3–4; Wilder 1952, pp. 53–57; Eves 1983, p. 474 [cited in Garciadiego 1992, p. 20]; Russell [1901] 1990; Whitehead and Russell 1910–1913, p. 1). There is now no doubt that this account is incorrect (Dreben in Bishop 1975, p. 517; Moore 1978, p. 323; Garciadiego 1992, chap. 2, and 1994, p. 630). Garciadiego (1992) argues convincingly that it is Russell alone who interpreted the discoveries of Cantor and Burali-Forti as paradoxes and that these were not known as paradoxes until after Russell published *The Principles of Mathematics* (Russell 1903).[1] Both the invention of modern logic and an extensive amount of work on the foundations of mathematics were accomplished before any of the paradoxes were discovered. What then, were mathematicians in the late nineteenth century trying to accomplish by the invention of algebraic logic, the invention of symbolic logic, the axiomatization of arithmetic and geometry, new definitions of number, and the large "formularies" that collected all of mathematics in one place?

Standard accounts of the origins of the philosophy of mathematics overemphasize epistemological concerns and ignore broader philosophical and disciplinary concerns, especially over the unity of mathematics, that were a strong motivating factor for the study of the foundations of mathematics and the development of logic. I do not claim that epistemological concerns did not exist—indeed, they appear to have been central to Russell and to have been widely held after the discovery of the set-theoretical paradoxes; however, reevaluating the development of scientific philosophy in the twentieth century requires a full understanding of issues that led to the study of the foundations of mathematics and the development of modern logic. Contextualizing the early work on the foundations of mathematics will lead to a different picture of the important role it played in the development of twentieth-century philosophy.

Consider Gottlob Frege, whose work is well known and who, of all those studying foundations, may be the mathematician most concerned with epistemological issues. Paul Benacerraf notes that there is something odd about the standard story of Frege's logicism without, however, reaching the same conclusions as I draw:

1. Of course there were some mathematicians who were critical of set theory prior to 1903, notably Kronecker.

> When I was young I was taught a number of fundamental propositions: Frege was the father of logicism. . . . I was told too that Frege had invented the logic that arithmetic was really only—or at the very least that he was the father of modern logic. Had I stopped to think, it might have occurred to me to question this all too happy coincidence of discovery and invention. At the very least, a decent interval should have been allowed to elapse between the discovery (invention) of the laws of logic and the further (?) discovery that they were just what had been needed to show that the basic laws of arithmetic had really been basic laws of logic all along. (Benacerraf 1981, pp. 41–42)

Far from being motivated by the paradoxes, Frege stopped his work on foundations as soon as the paradoxes became known. The main point of Frege's work was to show that mathematics is a priori, in order to fight against psychologism, not to prove the consistency of arithmetic or set theory (Sluga 1980, pp. 96–97). It also seems that Frege, like Giuseppe Peano, saw his own work as continuous with the rigorization of analysis that had taken place earlier in the nineteenth century (Demopoulos 1981). Frege's story shows that work on the foundations of mathematics must have been undertaken for other reasons, not in response to a concern over consistency due to the discovery of the paradoxes.

Although the paradoxes are mentioned by Alfred North Whitehead and Bertrand Russell, they constituted the third of three reasons cited as motivating the writing of *Principia Mathematica* ([1910] 1913, p. 1).[2] The other two reasons given are similar to those justifying the earlier work on foundations to be discussed in this article, finding the minimum number of primitive terms and axioms and the need to present mathematics in a clear language. Furthermore, although it is true that proving the consistency of mathematics was listed as an open problem in Hilbert's famous lecture of 1900 at the International Congress of Mathematicians (Hilbert [1901] 1932–35, p. 299, and 1902, p. 447), it is not the only open problem, nor was it as pressing for him in this lecture as it would become at the next International Congress (Hilbert 1904), after Russell's publication of the paradoxes, and much later when Hilbert developed his formalist program. Even then, the point of Hilbert's program (Hilbert [1917] 1932–35, 1921) was to show that one can prove

2. Rodríguez-Consuegra gives an excellent overview of Russell's motivations for studying foundations and many references to other literature (Rodríguez-Consuegra 1991, esp. pp. 185–89).

the consistency of arithmetic by "real" (i.e., finitary) means, not to assuage any fear that arithmetic could really be inconsistent.[3] In his pre-1903 writings, Hilbert was most concerned with finding the minimum number of primitive terms and axioms and with presenting mathematics in a clear language (e.g., 1921). There is no evidence to suggest that he thought that mathematics was in crisis or that some parts of mathematics could be false. Even when epistemological concerns are in evidence, there is clearly more to the story of the early work on the foundations of mathematics than merely an attempt to prove the consistency of arithmetic.

One alternative account of the interest in the foundations of mathematics is that developments in nineteenth-century geometry—such as the development of Boylai-Lobachevskii geometry as an alternative to Euclid's and especially the development of certain techniques in projective geometry—were central to the development of modern logic and to the study of the foundations of mathematics (Nagel [1939] 1979). Many commentators, however, assume that there was an epistemological crisis in geometry as well—that the certainty of geometry had been shaken by the development of non-Euclidean geometries in the nineteenth century and that geometers had to turn to something (logic!) to settle this crisis of proliferating geometries. Thus, the development of non-Euclidean geometry has often been seen as the result of a growing movement toward more rigor, more explicit definitions, more emphasis on proof, and the ferreting out of implicit assumptions, ending with modern logic and mathematical formalism (Bonola 1906, [1912] 1955; Coolidge [1940] 1963; Kline 1972). In these accounts, concern over foundational issues is thought to have led to the development of non-Euclidean geometries. For example, the famous 2,000-year-long discussion of the Parallel Postulate is seen as an attempt to prove the postulate rigorously, implying that geometers were uncertain as to whether Euclid's postulate was true. In his influential history of non-Euclidean geometries, Bonola said, "Even the earliest commentators on Euclid's text held that Postulate V. was not sufficiently evident to be accepted without proof, and they attempted to deduce it as a consequence of other propositions" (Bonola [1912] 1955, p. 3). Many historians of mathematics, including Morris Kline, have correctly noted that the aim in discussions of the Parallel Postulate was often to

3. Michael Hallett emphasizes that Hilbert had multiple aims and contrasts them with those of Frege (Hallett 1994, pp. 163–65). Hallett also notes that the unity of mathematics was an important issue for Hilbert in that he emphasized the use of the same logical methods in all branches of mathematics (Hallett 1994, pp. 170, 176–77; see also Detlefsen 1994, p. 8).

find the simplest statements among those that are intuitively true or to discover whether the assumption of the postulate is really necessary rather than to doubt its truth (Kline 1980, p. 78). Another frequently heard charge—that, while attempting to prove the Parallel Postulate, mathematicians were led over and over again into circular arguments by using implicit assumptions that are equivalent to it—is misleading at best, since some mathematicians, for example Wallis, explicitly replaced the Parallel Postulate with a different assumption. Some contemporary writers have gone so far as to claim that geometers came to a realization that they were dealing with a purely formal system and that terms that seemed meaningful and axioms that seemed to be assertions actually were neither (Contro 1976; Freudenthal 1957; Steiner 1964; Trudeau 1987, p. 162). Although there was extensive discussion of rigor in analysis and although this discussion clearly influenced Frege, Peano, and others, claims that modern logic solved a crisis in mid-nineteenth-century geometry are clearly false, because a formal view of mathematics did not exist until long after non-Euclidean geometries were developed and accepted.

Certain epistemological issues were central to debates over geometry in the nineteenth century—such as whether geometry is empirical and whether alternatives to Euclid are really possible; however, Coffa and others have shown that many issues were primarily semantic, that is, a clarification of the subject matter of mathematics (Coffa 1986, 1991). Without entering into the debates over the correct interpretations of his work in this article, Frege's comments about the shocking lack of rigor in arithmetic could be read in this light as well, since he complains that mathematicians are confused about the meaning of primitive terms (see esp. Frege [1884] 1953, 1914). In writings on the foundations of mathematics, the idea of a crisis is a common theme, though one that has often been immediately questioned (e.g., Bishop 1975 and the discussion that follows it). At the very least, many statements about "crises" in the historical development of mathematics are overstated, as will be shown below.

Both of these accounts ignore a third motivation for the study of the foundations of mathematics—unification. The nineteenth century saw a tremendous expansion of the fields of mathematics, leading to specialization and to the problem that mathematicians could not keep up with new developments or even understand all of mathematics. Work on foundations showed that diverse mathematical practices could be integrated into a single framework of axiomatic systems and that much of mathematics could be expressed in a single language. The logicist thesis went even further toward unity, claiming that all of mathematics could be viewed as a single large axiomatic system. The new framework was the product of an interdisciplinary coalition—a new group of mathematician-philosophers who regarded mathematics and logic as a single discipline, who were inspired by Leibniz's project of creating a universal language, and who shared a commitment to international scientific cooperation. These were scientific philosophers who saw themselves as breaking away from idealism and as maintaining a traditional link between philosophy and science, so the unity that they sought was quite different from the "monism" of the prevailing turn of the century neo-Hegelians.[4] Although unification is a perennial theme, mathematicians had specific reasons to be concerned about unity in the second half of the nineteenth century.

In analytic philosophy, Russell and Frege have dominated in historical accounts of the development of modern logic and of the early work on foundations. The development of logic and of the foundations of mathematics has been seen very much through Russell's eyes (or perhaps through the eyes of later standard interpretations of Russell), and for Russell it does often appear that epistemological issues are central. For example, when Russell produced a survey of Peano's work for *Mind* (written in October 1900, prior to his discovery of his paradox) he assured the editor G. F. Stout that the issues to be discussed are philosophical and indeed cast mathematical issues in an epistemological light. He described the rigorization of analysis as a crisis—a discovery that some of mathematics is false (Russell [1901] 1990, p. 352). Peano's work was described as a search for necessary and sufficient premises, a characterization that could be adequate, as long as it was made clear that no principles of mathematics were in doubt for Peano and none required further justification.

4. Grattan-Guiness claims that it was precisely this monism that impeded many philosophers' understanding of the new logic, so it is important to distance the argument here from that kind of unity (1986, p. 129). Of course, an inability to understand modern mathematics was another major impediment. We now know that Russell took longer than was once thought to break with idealism (Griffin 1991; Hager 1994; Hylton 1990; Rodríguez-Consuegra 1991), but whether he was influenced by monism is a question that cannot be addressed in this article. Boole may be another figure who advocated a different kind of unity than is discussed here, but there is debate over his views (cf. Laita 1980 and Smith 1983, p. 28). For recent reflections on the theme of unity, see Galison and Stump (1996). The idea of the unity of mathematics also continues well beyond the period discussed here. For example, Beaulieu points out that the most widely read and influential of Bourbaki's works strongly emphasized the unity of mathematics and that showing the unity of mathematics was an important part of their program (1994, p. 242; Bourbaki 1948, 1950). Even more recently, category theory has been seen by some as an intriguing unifying program (Mac Lane 1988).

Like Russell, Frege saw logic as a possible foundation of arithmetic (Frege 1880/81, p. 12). As important as Frege may be, the significance attributed to his work and to the effect of the paradoxes so strongly emphasized by Russell has "epistemologized" analytic philosophy's account of the early work on the foundations of mathematics. The idea of the unification of mathematics was a more widely acknowledged theme in turn-of-the-century mathematics than was the search for epistemological foundations. Some works that emphasized this theme, such as Peano's *Formulario*, were much more widely read than Frege's works, even though Frege made most of the major innovations in logic. I aim at a more balanced historical account and in my conclusion will return to Frege and Russell to discuss many viewpoints and influences they share with other mathematicians and philosophers interested in the foundations of mathematics at the end of the nineteenth century.

II. Unity

It is well known that mathematics, like so many other academic disciplines, went through a period of tremendous growth, specialization, and professionalization at the end of the nineteenth century (Cajori 1919, pp. 278–86; Gray 1987, p. 57, and 1992, pp. 237–38). It has also been widely noted that this progress led to the problems often associated with specialization. For example, Morris Kline remarks that:

> Accompanying the explosion of mathematical activity was a less healthy development. The many disciplines became autonomous, each featuring its own terminology and methodology. . . . From the nineteenth century on one finds mathematicians who work only in small corners of mathematics; quite naturally, each rates the importance of his area above all others. His publications are no longer for a large public but for particular colleagues. Most articles no longer contain any indication of their connection with the larger problems of mathematics, are hardly accessible to many mathematicians, and are certainly not palatable to a large circle. (Kline 1972, p. 1024)

By the turn of the century, the dramatic expansion of mathematics in the nineteenth century was widely presented as problematic:

> The extraordinary development of mathematics in the last century is quite unparalleled in the long history of this most ancient of sciences. Not only have those branches of mathematics which were taken over from the eighteenth century steadily grown, but entirely new ones have sprung up in almost bewildering profusion, and many of them have promptly assumed proportions of

> vast extent. (Pierpont 1905, 1:474, as cited in Moritz [1914] 1958, p. 110)
>
> The science [of mathematics] has grown to such vast proportion that probably no living mathematician can claim to have achieved its mastery as a whole. (Whitehead [1911] 1958, p. 188, as cited in Moritz 1914, p. 119)

The growth and disunity of mathematics were frequently discussed, and various possible solutions were put forth. At the end of his famous lecture on open problems in mathematics at the Paris Congress of 1900, Hilbert suggests that the unity of mathematics needed to be defended and that logic would come to the rescue:

> The problems mentioned are merely samples of problems, yet they will suffice to show how rich, how manifold and how extensive the mathematical science of today is, and the question is urged upon us whether mathematics is doomed to the fate of those other sciences that have split up into separate branches, whose representatives scarcely understand one another and whose connection becomes ever more loose. I do not believe this nor wish it. Mathematical science is in my opinion an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts. For with all the variety of mathematical knowledge, we are still clearly conscious of the similarity of the logical devices, the relationship of the ideas in mathematics as a whole and the numerous analogies in its different departments. (Hilbert 1902, p. 478)

Hilbert was not alone in taking logic to be fundamental nor in his claims that it could provide unity.

> The development of abstract methods during the past few years has given mathematics a new and vital principle which furnishes the most powerful instrument for exhibiting the essential unity of all its branches. (Young and Jackson 1911, p. 225, as cited in Moritz 1914, p. 117)

Still, the unity of mathematics was not obviously accomplished at the turn of the century. For example, the question of whether Whitehead could sustain the unity that he had achieved in *A Treatise on Universal Algebra* ([1898] 1960) in a projected second volume was raised in a review that appeared in *Nature:*

> It is, in fact, a comparative study of special algebras, exclusive of ordinary algebra, the results of which are taken for granted

> throughout. Such an undertaking has necessarily involved a very great deal of time and labour; for, in order to carry it out with any degree of success, it is needful, not only to master each separate algebra in detail, but also to adopt some general point of view, so as to avoid the imminent risk of composing, not one work, but a bundle of isolated treatises. Mr. Whitehead has, happily, overcome this difficulty by viewing the different algebras, in the main, in their relation to the general abstract conception of space. Whether this plan can be consistently followed throughout may be open to question: it certainly works very well in this first volume, the keynote of which is Grassmann's Extensive Calculus. (Mathews 1898, p. 384)

Much later, *Principia Mathematica* remained incomplete because Whitehead never wrote the projected fourth volume. Grattan-Guiness has recently emphasized the question of whether the unity of mathematics was ever proven by the logicists (1985, pp. 73–74). Furthermore, as the quotation just cited shows, although there was agreement on the need for fundamental principles and concepts to establish the unity of mathematics, there was some disagreement about what the fundamental concepts of mathematics were. For example, in 1883 Cayley claimed that "the notion, which is really the fundamental one (and I cannot too strongly emphasize the assertion), underlying and pervading the whole of modern analysis and geometry, is that of imaginary magnitude in analysis and of imaginary space in geometry" (Cayley 1883, p. 434; see also Pierpont 1905, p. 474). Twelve years later, Lie would say, "In our century the conceptions substitution and substitution group, transformation and transformation group, operation and operation group, invariant, differential invariant and differential parameter, appear more and more clearly as the most important conceptions of mathematics (Lie 1895, p. 261, as cited in Moritz 1914).

Even earlier, in his famous Erlanger Program of 1872, Felix Klein had argued that unity would be obtained through projective geometry and invariant theory (see Gray 1992, p. 238, and 1986, chap. 6; Pyenson 1983). By the turn of the century, Klein returned to the theme of unity, still seeing the massive expansion of nineteenth-century mathematics as a problem and still hoping that unity could be achieved through combining the results of many areas of mathematics "into certain general conceptions" ([1893] 1896, p. 134). He now saw "function" as the basic concept that applies everywhere.

The existence of many unifying programs provides evidence in favor of my claim that reconstructing the unity of mathematics was a central project of late nineteenth-century mathematicians, even if Klein and other mathematicians sometimes changed from one unifying program to another. For example, although Poincaré supported Lie in taking the concept of a group to be fundamental, he changed from a group theoretical basis to a topological one after reading Hilbert (Poincaré 1904; Johnson [1979] 1981; Stump 1996). I claim that much of the work on foundations and the formalization of mathematics—the axiomatization of mathematical theories, the use of explicit definition, and the reduction of the number of primitive terms and of axioms to a minimum—was in response to the problem of growth and specialization in mathematics during the nineteenth century. Many mathematicians searched for unifying basic concepts and attempted to reduce the number of primitive terms and axioms to a minimum (Peano 1894*a*, p. 41; Hilbert 1921).

There is little doubt that logic and formalization won the struggle to be taken as fundamental in geometry and elsewhere and that they helped mathematicians to reconstruct the unity of at least much of mathematics. By 1913, all known geometries were shown to fit into Hilbert's framework (Gray 1994, p. 906). A recent assessment highlights Hilbert's spectacular success in both reconstructing the unity of mathematics from an axiomatic standpoint and gaining the approval of other mathematicians.

> The modern form of the axiomatic method—undoubtedly, under the influence of Bolyai and Lobachevsky's work—has taken shape in the publications of Hilbert. The method gained the approval of contemporary mathematicians within a short time, and this led first to the renewal of algebra and then to the rebuilding of all mathematics. In the proof of a theorem, only a few features of the objects of mathematical research play a role; so the proof can be applied to other objects also having these features. Using this idea, the proof of an assertion is performed without specifying the objects involved; instead, those properties of the objects on which the proof is based are listed as axioms. A theorem proved in this way will be valid for all objects which satisfy the axioms.
>
> Working in this spirit, several fields of mathematical problems considered previously to be separate and remote from one another could be treated alike as particular cases of a more comprehensive problem. On the other hand, constructive intuition led to the discovery of rather interesting special cases, and of new problems outside even the enlarged field of questions.

> As the axiomatic method spread to all territories of mathematics, classic limits of subdivision dimmed. New classification and reordering of the basic mathematical notions were directed by the ideas of structure theory. Mathematics has thus become more uniform. The deepening of abstraction, the widening of generalization promoted science further and further in exploring reality. (Kárteszi 1987, pp. 218–19)

The most important reason for reconstructing the unity of mathematics seems to have been the perceived need to compare results in one area with results in another. The project required a total rewriting of mathematics into a single language and necessitated the development of techniques that could be used in many, or even all, branches of mathematics. Whitehead's description of his project in his early work *A Treatise on Universal Algebra* shows very clearly what is involved; he would find general methods to connect disparate branches of mathematics, each represented in a section of his book. "The various parts of this volume have been systematically deduced, according to the methods appropriate to them here, with hardly any aid from other works. This procedure was necessary, if any unity of idea was to be preserved, owing to the bewildering variety of methods and points of view adopted by writers on the various subjects of this volume" (Whitehead [1898] 1960, p. x).

The *Formulario* of Peano and his school is emblematic of this grand reworking of all of mathematics into a single format. Setting out to write down all of the truths of mathematics in one place, Peano saw his project as a continuation of Leibniz's; all of mathematics would be written in a universal language and would be amenable to proof. In a summary of his work on mathematical logic to date (1891), Peano says:

> One of the most notable results reached is that, with a very limited number (7) of signs, it is possible to express all imaginable logical relations, so that with the addition of signs to represent the entities of algebra, or geometry, it is possible to express all the propositions of these sciences.

In a note, Peano continues,

> It thus results that the question proposed by Leibniz has been completely, if not yet perfectly, resolved. (Peano 1973, p. 154)

The most important issue for Peano, beyond his belief in universal artificial languages, is that all of the branches of mathematics should be understandable and that it should be possible to compare the results in one area to those in another (Peano 1901, p. v, and 1894*b*, p. 52; Galdeano 1897).

III. Geometry

The development of new geometries, especially non-Euclidean geometries, has been taken as marking a real revolution in mathematics—indeed, a crisis that led to work on the foundations of mathematics. Although Bolyai, Lobachevskii, and Gauss independently developed the first professed non-Euclidean geometry around 1830, the work of Bolyai and Lobachevskii received little attention, and Gauss's remained unpublished. In the late 1860s, Beltrami and Klein provided interpretations of Bolyai-Lobachevskii geometry (BL), and Riemann's work became more widely known. By the 1870s, general expositions of non-Euclidean geometries appeared. Ironically, there were no critiques of non-Euclidean geometries until after 1870, immediately after the work of Riemann, Beltrami, and Klein had finally put BL on a firm foundation (Sommerville [1911] 1970). It was not until Riemann and Helmholtz gave a philosophical interpretation of the existence of non-Euclidean geometries, pointedly arguing that their existence refuted Kant's philosophy, that non-Euclidean geometries were widely discussed and debated (Nowak 1989). At the turn of the century, Peano and Hilbert wrote new axiom systems for geometry.

The debate over non-Euclidean geometries after 1870 is not indicative of a foundational crisis as is suggested in many standard histories of mathematics. Indeed, the consistency of BL and the unprovability of the Parallel Postulate was already accepted by mathematicians before the work of Riemann, Beltrami, and Klein, even though everyone admits that the work of Bolyai and Lobachevskii is woefully inadequate from a logical point of view. Bolyai and Lobachevskii assumed the acute angle hypothesis and derived consequences; they developed a non-Euclidean geometry by proving theorems but had little to say about consistency. For example, in his first paper, Lobachevskii claimed that his geometry was consistent because he obtained its trigonometric formulas from the corresponding formulas of spherical geometry by multiplying the sides of a triangle by an imaginary unit, but all he proved is that these formulas are consequences of the assumptions of BL. To prove consistency, all the formulas of BL must be shown to be consequences of some other geometry, a proof which Lobachevskii did attempt in a later work, without arriving at a complete proof (Rosenfeld 1988, p. 227–28).

The argument given in textbooks of the time was on an even less sound logical footing. It was simply stated that the attempt to prove the Parallel Postulate had gone on too long, and that the repeated lack of success shows that it is impossible (a traditional argument, often mentioned in the French literature, that goes back to d'Alembert and Klügel). Before the work of Helmholtz, Beltrami, and Klein had appeared, Jules Hoüel, the French mathematician who was as responsible as Riemann and Helmholtz for making the existence of non-Euclidean geometries known by translating all of the principal texts into French and tirelessly advocating them, claimed that it was already clear that the Parallel Postulate was independent and that BL was consistent: "Research which is already old, but which has gone unnoticed until recently, has placed beyond doubt that the proof of Euclid's eleventh axiom (our fourth axiom) cannot be deduced from the preceding axioms. This discovery was made around 1829 by two geometers, Lobachevskii and J. Bolyai" (Hoüel 1867, p. 72).[5] Manning's textbook of 1901 takes exactly the same point of view, and it is most striking that he does not mention the models of BL that had been given by Beltrami, Klein, and Poincaré (Manning 1901). There is no evidence that Riemann, Beltrami, or Klein was responding to a "crisis in the foundations of geometry." Quite to the contrary, mathematicians seemed to have accepted the consistency of BL geometry and the impossibility of proving the Parallel Postulate before the work of Riemann, Beltrami, and Klein had put the non-Euclidean geometries on a sound logical footing. Therefore, it seems clear that something other than the invention of modern logic is behind the acceptance of the consistency of BL. Geometers in this period did not treat geometries as pure uninterpreted axiomatic systems, and yet they accepted the consistency of BL. The percentage of geometrical works devoted to the theory of parallels declined sharply after 1870 and continued to decline through the last decades of the nineteenth century (see table A1).

From 1870 on, heated philosophical discussions ensued, revolving around the issue of which geometry is true and how geometry is known; however, this discussion can certainly be considered continuous with earlier philosophical discussion of mathematics and of a priori knowledge in general. The fact that empiricism in geometry had already been championed by Mill, for example, shows that there is more continuity than might be expected in the philosophical debate over Euclid, and in that sense, the revival of empiricism by Riemann

5. "Des recherches déjà anciennes, mais qui ont passé inaperçues jusqu'à ces derniers temps, ont mis hors de doute que la démonstration de l'axiôme 11 d'Euclide (notre axiôme IV) ne peut pas se déduire des axiômes précédents. Ces recherches ont été faites vers l'année 1829, par deux géomètres, Lobatchewsky and J. Bolyai." All translations from original sources are mine.

and Helmholtz can hardly be seen as revolutionary. A revolution did occur in philosophy, however, when formalism in mathematics took hold, but, interestingly, many of the major actors in that movement did not see a connection to geometry. Even in mathematics proper, the development of spherical geometry was not considered to be revolutionary at all, because it was not seen to conflict with Euclid. In fact, spherical geometry conflicts with Euclid every bit as much as BL (Gray [1979] 1989, p. 71). It was only when Bonola and other historians started drawing philosophical consequences from the development of geometry in the nineteenth century that the revolution that had already taken place could be perceived.

The points raised above cast doubt on the claim that geometers used logic to end a crisis in the 1870s. In a classic and widely cited paper, Nagel reversed the order of the crisis account and argued that the development of analytic techniques in geometry provided the means for developing formal axiomatics at the turn of the century, basically by showing that geometric primitives are uninterpreted or reinterpretable and that the abstract relationships given in the axioms remain the same under different interpretations ([1939] 1979; see also Bourbaki 1970 and Freudenthal 1974, both of whom agree with Nagel). However, it is far from clear to what extent the development of nineteenth-century geometry influenced the modern conception of logic and mathematics. Indeed, we have already seen that another standard account gives the discovery of paradoxes in set theory, not the discovery of non-Euclidean geometries, as the motivation for formalization and the study of the foundations of mathematics. Frege maintained, notoriously, that there was a profound difference in the grounding of geometry and that of arithmetic (Bynum 1993), and he did not claim that geometry is part of logic. Furthermore, Whitehead and Russell did not mention geometry as a motivation for their study of the foundations of mathematics in *Principia Mathematica*, despite the fact that Russell was well aware of the development of non-Euclidean geometries and had written quite a lot about its philosophical implications. Of course, by then Russell had the paradoxes as a motivation. Thus, those studying the foundations of mathematics immediately after the turn of the century did not see a revolution in geometry as the basis for their new work. To this day, textbook justifications of the study of the foundations of mathematics do not include geometry. By striking contrast, the contemporary undergraduate textbook presentation of geometry does include a heavy dose of formal axiomatic methods.

Both the crisis account and Nagel's account strongly link the history of the discussion of the Parallel Postulate and discovery of non-

Euclidean geometries with the development of mathematical logic and discussion of the foundations of mathematics. This linkage is also stressed by contemporary geometry textbooks that treat the discussion of the Parallel Postulate and the consistency of BL as a historical example of a problem that can be solved with modern axiomatics and logic, with the geometry sometimes getting less treatment than axiomatics (Meschkowski 1965; Greenberg 1980; Trudeau 1987; Perry 1992). Nothing is necessarily wrong with this approach, as long as it is not claimed to be historical; in fact, it is interesting and important to note that the development of non-Euclidean geometries should be taken to be such a good pedagogical topic for introducing axiomatics, but it is equally noteworthy that earlier textbooks developed systems of non-Euclidean geometry without formal proofs of consistency (Manning 1901; Coolidge 1909).

The redescription of the development of non-Euclidean geometry allowed by a formal conception of mathematics became the standard, if whiggish, history. It is well known that histories of the sciences are often used to legitimate new sciences or revolutionary changes (Graham, Lepenies, and Weingart 1983), and the use of the history of the Parallel Postulate to justify the study of the foundation of mathematics and formal logic in elementary geometry textbooks is surely an example of such legitimation, especially since the original discussion of the foundations of mathematics did not include discussion of geometry as a justification. Furthermore, in this case, history, as well as logic, helped formalize mathematics. The narrative reconstruction of the development of geometry in the nineteenth century is part of the content of science just as much as the logical techniques are. By this I mean quite straightforwardly that the history of non-Euclidean geometry is part of what is taught in standard university courses in geometry and that a connection between the history of the discussions of the Parallel Postulate, the need for formal axiomatic systems, and the demand for rigorous proof all form part of this standard curriculum.

There is another positive side to this narrative reconstruction that should be mentioned, however, since it shows the power of the study of foundations occurring at the turn of the century. Diverse mathematical practices could be integrated into a new framework and justified by powerful methods in compelling ways. The consistency of BL could be proven with new rigor in the new logical framework, and a new narrative reconstruction of what had happened in geometry in the nineteenth century could be written. So, both a new logical and a new historical unity of geometry was achieved by the turn of the century. Bonola's work contributed to the broader unity of mathematics movement by showing a unified logical development of geometry. Histories do not, however, always show the unity of a science or that later developments in a science incorporate what came before. As Rachel Laudan has shown in a survey of early histories, contrary to what would be expected, histories of science were only rarely used to show the unity of science (1993, p. 21). Furthermore, the situation in nineteenth-century geometry that I have been describing is unique. The proliferation of geometries and the demands of disciplinary specialization at the end of the nineteenth century motivated the mathematicians' quest for reconstructing the unity of mathematics.

The two early influential histories of the non-Euclidean geometries are those of Stäckel and Engel, published in German in 1895, and of Bonola, published in Italian in 1906, expanded and translated into German in 1908, reprinted in 1919 and 1921, translated into English in 1912, reprinted in 1938 and 1955, and translated into Spanish in 1945. Although Stäckel and Engel did most of the original historical work, Bonola's book was much more widely disseminated and has become the standard history of the development of non-Euclidean geometry. A statement from Carruccio, if taken literally and as if he were speaking for the whole mathematical community, rather than as it was intended, could serve to highlight the impact of Bonola's history: "When nothing is said to the contrary, we refer always to Bonola" (Carruccio [1951] 1964; table A2 below).

What was the motivation for writing, translating, and publishing the history of non-Euclidean geometries at the turn of the century? There seem to have been two main motivations: The first was to convince philosophers that non-Euclidean geometries are consistent, an issue that the mathematical community had already settled by 1870. The second was to draw philosophical lessons from the development of non-Euclidean geometries and to give teachers of mathematics new tools for pedagogy. Both of these motivations required that the development of non-Euclidean geometries be described in a form that would be useful. It appears that describing the developments as a revolution occurring in response to a crisis was just what was needed to show the limits of the rational intuition used by classical mathematicians and philosophers and to convince students that they should learn how to express themselves in symbolic terms and to formulate rigorous proofs.[6]

6. Peano had a notorious teaching record. He claimed that the *Formulario* could serve as a textbook, but his actual use of it in the classroom apparently led to a large number of student complaints (Kennedy 1980, pp. 100–102).

In his introduction, Stäckel affirms that he wrote his history partly as a conscious attempt to convince philosophers. "If we have so far directed ourselves to the mathematicians, we would also like to bring our book to the attention of philosophers, for the theory of parallels is closely connected with various foundational problems of philosophy; as Gauss has said, the central question touches directly upon Metaphysics" (Stäckel and Engel 1895, p. vi).[7] Bonola, on the other hand, says very little about his explicit motivation for writing his history of the non-Euclidean geometries. Nonetheless he does assert that it was written in response to "the interest felt in the critical and historical exposition of the principles of various sciences" ([1912] 1955, p. vii). The term "historical-critical" usually refers to biblical exegesis and also dates, for example, to Bayle's famous encyclopedia (Bayle 1695–1702; Krentz 1975). It started appearing in histories of science and of philosophy in the mid-nineteenth century and was fairly common by the end, with Mach's *Science of Mechanics* (1891) being the most famous example. At the turn of the century, the *Bulletin of the American Mathematical Society* was subtitled "A Historical and Critical Review of Mathematical Science." The term may have been so widely used and commonplace that it lost its specific meanings; however, it is clear that "historical-critical" referred to an analytic stance by which the reliability of sources was to be examined and by which historical texts were to be understood in their context. The historical-critical method is associated with the Enlightenment and, in the case of Bayle, with strict empiricism, which may explain Mach's interest in the term. Although it is unclear exactly what Bonola or Mach had in mind in using these terms, it is clear that both had the intention of using modern historical methods.

Enriques, in the preface he wrote for the English translation of Bonola's book, is considerably more revealing than Bonola was himself.

> It seems to me that this account, although concerned with a particular field only, might well serve as a model for a history of science, in respect of its accuracy and its breadth of information, and, above all, the sound philosophic spirit that permeates it. The various attempts of successive writers are all duly rated according to their relative importance, and are presented in such a way as to bring out the continuity of the progress of science, and

7. "Haben wir uns bis jetzt an die Mathematiker gewendet, so möchten wir doch auch die Philosophen auf unser Buch aufmerksam machen, denn die Parallelentheorie steht mit verschiedenen philosophischen Grundproblem in enger Verbindung, streift doch, wie Gauss sich ausdrückt, der Fragepunkt unmittelbar an der Metaphysik."

> the mode in which the human mind is led through the tangle of partial error to a broader and broader view of truth. This progress does not consist only in the acquisition of fresh knowledge, the prominent place is taken by the clearing up of ideas that it has involved; and it is remarkable with what skill the author of this treatise has elucidated the obscure concepts which have at particular periods of time presented themselves to the eyes of the investigator as obstacles, or causes of confusion. . . . May his devotion stimulate others to pursue with ideals equally lofty the path of historical and philosophical criticism of the principles of science! Such efforts may be regarded as the most fitting introduction to the study of the high problems of philosophy in general, and subsequently of the theory of understanding, in the most genuine and profound signification of the term, following the great tradition which was interrupted by the romantic movement of the nineteenth century. (Bonola [1912] 1955, pp. iii–iv)

The references to the eighteenth-century view of progress and the unity of science are particularly significant because they highlight the fact that those studying the foundation of mathematics had found a way to unify mathematics, something that had been lost in the great expansion of mathematics during the nineteenth century and in the "romantic" attacks on science in the late nineteenth century. Part of this gain in unity, however, comes from Bonola's reformulation of geometers' ideas in axiomatic terms and his highlighting of the issues seen as important at the turn of the century. Gray's criticism of the standard history of non-Euclidean geometries—Bonola's work and that of others—matches Enriques' assessment very well. Gray says that the main problem with the standard account is its tendency to regard the entire history of the Parallel Postulate as comprising a single issue—the foundations of geometry (Gray [1979] 1989, pp. 169 and 149–150; 1987, p. 57). Thus, Bonola's emphasis on the foundational issues, which Gray justly criticizes as whiggish, is praised by Enriques for clarifying the issues and providing a unified framework for evaluating the history of geometry.[8] There is no doubt that Bonola's book has been read from Enriques onward as contributing to the foundations of geometry.

8. It is worth pointing out that Bonola's whiggism seems quite self-conscious. He "steps out" of purely chronological historical presentation to introduce a section on "Hypotheses equivalent to Euclid's Postulate" (Bonola [1912] 1955, pp. 118–21), where he introduces a contemporary definition of logical equivalence, and in an appendix on "The [logical] Independence of Projective Geometry from Euclid's postulate" (pp. 227–37). As Gray notes, the appendix did not appear in the original 1906 Italian edition.

It is also clear that these influential histories of the non-Euclidean geometries were written and published with pedagogy in mind. Here Carslaw, the English translator of Bonola's text, speaks for the mathematical community. "Recent changes in the teaching of Elementary Geometry in England and America have made it more than ever necessary that those who are engaged in the training of the teachers should be able to tell them something of the growth of that science" (Bonola [1912] 1955, p. v). There is also no doubt that the pedagogical literature was influenced by the standard historical account of the development of the non-Euclidean geometries and that Bonola's account was influenced by the study of the foundations of mathematics. As mentioned earlier, the standard undergraduate course in geometry now includes a strong dose of formal axiomatics, but it also includes a very surprising amount of history, far more than is typically found in mathematical textbooks on other topics. Increasingly, formal logic and axiomatics are being introduced independently of geometry in newer textbooks so that these new texts may serve a dual function as an introduction both to mathematical logic and to geometry. At the same time, the historical accounts are being left out of these textbooks, so that the linkage forged by mathematicians writing history at the turn of the century between mathematical logic and geometry is now being obliterated (see table A2).

IV. Conclusions

Five themes recur in the works of those involved in the foundations of mathematics and the development of new logics, themes that tie this work to broader movements in science and philosophy. The foundations of mathematics and the new logics were the product of an interdisciplinary coalition, a new group of mathematician-philosophers. Frege, Whitehead, Peano, Russell, Couturat, and, to a lessor extent, Boole, Schröder, and Hilbert partially share a set of interconnected influences and general philosophical outlooks (enough to maintain a family resemblance): the strong influence of Leibniz, logicism, interdisciplinarity (taking degrees in both philosophy and mathematics or publishing in both fields), belief in the fundamental unity of mathematics, and internationalism. Many of these ideas resemble those later adopted by the Vienna Circle and logical empiricists.

A. The Leibnizian Revival

Peano, Couturat, Frege, and Russell all cited Leibniz frequently. At the turn of the century, Russell and Couturat independently wrote pioneering works on the logic of Leibniz (Couturat [1901] 1969; Russell

1900), and Couturat discovered and published previously unknown logical writings of Leibniz, the so-called Hanover manuscripts (Couturat 1903). The fact that Leibniz was so widely cited at the turn of the century is somewhat surprising, in light of the fact that his logical writings had been ignored earlier. For example, writing in 1854, Boole mentions practically everyone except Leibniz, providing evidence for the contention that only at the turn of the century was Leibniz considered the premier logical precursor (Boole [1854] 1958). We find in Leibniz an element of each of the themes highlighted here. At least in the work of the new Leibniz discovered at the turn of the century, logic plays a central role, echoing logicism. As the coinventor of the calculus and the developer of a relational theory of space that rivaled Newton's absolute theory and other scientific work, Leibniz certainly represented scientific and, especially, mathematical philosophy. Compared to claims of the most famous advocate of "universal harmony," the unity of mathematics seems a very modest claim, indeed. And finally, Leibniz invented the idea of a universal language and of a universal calculator, believing that all disputes could be settled by rational means. Perhaps advances in logic made Leibniz's claims appear more tenable, or perhaps those working on logic simply used Leibniz as an emblem, finding all of the themes they supported already personified in him. At the turn of the century, Leibniz was suddenly an oft-cited precursor, becoming a model for what it was to be a philosopher.

B. Logicism

Those working on algebraic logic, such as Boole, Peirce, Jevons, Venn, Schröder, and Whitehead, saw themselves as making logic more mathematical. Conversely, according to the standard reductionist reading of Frege and Russell's logicism, mathematics (for Frege only arithmetic) should be seen merely as a branch of logic. So, is logic a branch of mathematics, or is mathematics a branch of logic? Although Peano agreed that mathematics should be more logical, he maintained that mathematics and logic are separate disciplines. Indeed, Peano's main critique of algebraic logic is that mathematical symbols were used for logical notions. His great advance over algebraic logic was to introduce special logical symbols and to thereby make distinctions that had been ignored previously, such as the distinction between the various uses of "is" and the distinction between set membership, proper inclusion, and improper inclusion (Grattan-Guinness 1984, pp. 7–8). On the other hand, Couturat sometimes equated logic and mathematics, creating either a grand synthesis of the two traditions or else a terrible confusion (Couturat 1904, pp. 1046–47; 1905, p. 5).[9] Even if logicism no longer seems a viable project, all of these authors strongly linked logic and mathematics, forging links that endure to this day.

C. Scientific Philosophy

The philosophers working on the foundations of mathematics were scientific philosophers who saw themselves as breaking away from idealism and as maintaining a traditional link between philosophy and science. Grattan-Guiness has said that there were very few in the period who were truly both mathematician and philosopher, listing only Boole, De Morgan, Peano, and Whitehead and excluding both Russell and Frege as well as Couturat (Grattan-Guinness 1988, p. 79). It is true that rather than create original mathematics, Couturat gave a philosophical defense of Cantor's set theory and that, despite evidence to the contrary (Boolos 1994), Russell is often thought to have contributed advances only in logic proper—but, significantly, they each took degrees in both mathematics and philosophy. Pierre Boutroux gave a very telling description of Couturat as "conforming to the classical tradition," given that he had completed his philosophical education by taking a degree in a science (1892, quoted in Sanzo 1983, p. 70; see also Giuculescu 1983). Surprisingly, this suggests that there was only a very brief period in the nineteenth century when it would have been unusual for a philosopher to study a science. Russell, of course, more than anyone, became the model for philosophers of mathematics, having written his dissertation on geometry before turning to arithmetic and set theory and having consciously based his philosophy on results from modern logic.

Frege's critique of psychologism (clearly a philosophical theory of mathematics) was very influential, especially through Edmund Husserl. Furthermore, his paper "On Sense and Reference" became a seminal work in the philosophy of language. Although his philosophical papers were narrow in scope, they had a large impact on philosophers in the twentieth century. Peano and his collaborators were involved in foundational issues, and Peano did attend the International Philosophy Congresses in 1900 and 1904, even though he apparently took practically no interest in philosophical issues, per se. Hilbert believed that both philosophical and mathematical investigations were neces-

9. Bowne notes this feature of Couturat's logicism, citing different passages (1966, p. 52).

sary to complete his program, taking quite an active part in supporting the hiring and promotion of scientific philosophers at Göttingen, even if his own works were not philosophical (Peckhaus 1994). In spite of these limits and qualifications, the level of cooperation between philosophers and mathematicians at the turn of the century was remarkable and unprecedented. The process of developing this interdisciplinary coalition led to the creation of the philosophy of mathematics as a specialization in philosophy. As it was later developed by the Vienna Circle and logical empiricists, knowledge of modern logic and the foundations of mathematics became the cornerstone of scientific philosophy.

D. The Unity of Mathematics as a Project

The quotations given in Section II above demonstrate that, by the late nineteenth century, many mathematicians felt the need to unify mathematics, but how do Frege and Russell stand on this issue? Since a strong form of the unity of mathematics follows from logicism—all of mathematics could be seen as one large axiomatic system—one might think that reconstructing the unity of mathematics could be taken as central to their project as well. However, the unity of mathematics is not a theme found in their work, although Frege does claim in the preface of the *Begriffsschrift* that it is possible to write all of mathematics in a formula language such as his (Frege [1879] 1972, pp. 105–6). Frege and Russell seem to have had other motivations and did not often express a need for unity. The fact that the unity of mathematics follows from logicism is not a sufficient reason to attribute to them an interest in the unity of mathematics (May 1975).

Whitehead, Peano, Couturat, Hilbert, and Boole all saw the unity of mathematics as very important and cited the reconstruction of the unity of mathematics as one of the aims of their works. I noted above that it is questionable whether the unity of mathematics has ever been proven; however, since formal axiomatics provided the clearest and most powerful framework for proving the consistency of BL and of other axiomatic theories, it did allow a redescription of the development of non-Euclidean geometry, a description that has become the standard account.[10] The existence of this standard account shows what the study of foundations achieved. Only the study of the foundations

10. It has been noted that axiomatization tends to come late in the development of a field: See the consensus that developed in Sacks's discussion (1975, p. 528).

of mathematics with formal axiomatic methods allowed a coherent cumulative picture of nineteenth-century geometry to be drawn. The nineteenth-century revolution in geometry did have profound implications for twentieth-century philosophy; the logical positivists took all scientific theories to be expressible as uninterpreted axiomatic systems then given a physical interpretation through bridge principles. Alberto Coffa argues convincingly that the linguistic conventionalism of Carnap and Wittgenstein came straight out of the development of a formal conception of geometry (Coffa 1986), though I would argue that the formal conception of geometry that influenced the logical positivists derives from a particular reading of Poincaré and Einstein's physics, not directly from pure mathematics. In the early twentieth century, the logical positivists used modern mathematical logic to carry out a central part of their project of developing a scientific philosophy. It allowed them to eliminate intuitive elements from mathematics and science in general and to replace them with explicit rules of construction.

E. International Scientific Cooperation and Universal Languages

The turn of the century was a time of widespread international scientific cooperation, marked by a remarkable amount of travel by scientists to international conferences and World's Fairs, by the development of new organizations, and by the work on standards (Lyons 1963, pp. 223–45). Many of the above quotations are from the International Mathematical Congresses that began with sessions as part of the World's Fair in Chicago in 1893 and continued more formally in 1897 in Zurich, 1900 in Paris, 1904 in Heidelberg, 1908 in Rome, 1912 in Cambridge, and so on (Albers, Alexanderson, and Reid 1986). The first formal congress was purposely held in neutral Switzerland, after which the site was rotated among countries. The First International Philosophy Congress was organized in 1900 in Paris under the auspices of the *Revue de Métaphysique et de Morale,* and Paris was also the site of another World's Fair that year, one of the biggest ever. As is well known, Russell, Couturat, Peano, and others attended both the mathematics and philosophy congresses in Paris. Russell met Peano for the first time at the Paris Congresses, asked for all his works, adopted his notation, and immediately began working on symbolic logic within Peano's system.

The bibliographic projects started in conjunction with the World's Fair in Paris in 1889 (Répertoire Bibliographique des Sciences Mathématiques 1893) and by the Royal Society in London, which was both broader in scope and more broadly international (Provisional International Committee [1901] 1968), provide further examples of international scientific cooperation. The goal of both organizations was to publish a complete list of mathematical literature, as well as to develop classification schemes that would provide a degree of organizational unity to mathematics. The role of these publications in unifying mathematics and mathematicians was recognized at the time (Galdeano 1897). International mathematical cooperation reached its zenith with the congresses and with the joint publication in German and French of the *Encyclopedia of Mathematical Knowledge* (Meyer 1899–1916; Molk 1904–16) but disintegrated during World War I, as did cooperation in many international scientific organizations (Crawford 1992; Schroeder-Gudehus 1990). The *Encyclopedia* was never completed, and congresses did not take place in 1916 or from 1937 to 1949. When the congresses resumed after World War I, they became politicized, with most mathematicians from Germany and other central powers excluded from Strassbourg (1920) and Toronto (1924). When the congresses were again opened to all mathematicians in 1928 (Bologna), there was a call by some German mathematicians to boycott them. Hilbert personally led a delegation of German mathematicians to the congress in order to break the boycott, speaking in favor of internationalism in mathematics during his address (Albers et al. 1986, p. 20; Reid 1970, p. 188).

Peano and Couturat were both strong supporters of artificial universal languages. At the first mathematical congress in Zurich, Couturat set up a commission to study the use of artificial auxiliary languages in mathematics. Although the need to find a common language was taken quite seriously by mathematicians at the turn of the century, only a few went as far as Peano and Couturat. Peano published his works in Latin and French, as well as Italian, in a conscious attempt to be more internationalist, and invented the artificial auxiliary languages Latino Sine Flexione and Interlingua for his publications. Couturat was involved in the Esperanto movement but broke off to support another artificial auxiliary language, Ido, that he advocated with his journal *Progreso* (Lyons 1963, pp. 208–15). The other mathematician philosophers mentioned did not follow Peano and Couturat in their support of auxiliary languages, but Russell, at least, did become an internationalist in a political sense. In an exchange of letters around 1900, Couturat took Russell to task for his support of the Boer War and for not supporting free trade, a position that at the time was equated with internationalism. Russell very quickly came around to Couturat's position; indeed, free trade became the first of the many controversial liberal causes that Russell would champion (Rempel 1979; Russell 1985). In several letters to Couturat, Russell also acknowledged that an international language for science would be useful and expressed support for Couturat's efforts.[11]

The turn of the century represented an end and a new beginning, a time of grandiose displays of science and technology and of antiscience backlash (Paul 1968). There were mobs at the sight of an airplane; larger and larger crowds at the world's fairs; monumental projects such as railroads, tunnels, bridges, electrification, and plumbing; international scientific cooperation; celebration of colonialism; "scientific study of primitive people"; and great claims of progress. It also seems that the turn of the century marked a return of some philosophers to science. The cooperation between philosophers and mathematicians at the turn of the century led to the creation of the philosophy of mathematics as a specialization in philosophy, and knowledge of modern logic and the foundations of mathematics became the cornerstone of scientific philosophy.

Standard histories of mathematics and of analytic philosophy maintain that work on the foundations of mathematics was motivated by a crisis such as the discovery of paradoxes in set theory or the discovery of non-Euclidean geometries. I have argued that the study of the foundations of mathematics was motivated in part by a compelling interest in the unification of mathematics on the part of mathematicians who, in response to the fragmentation and rapid expansion of nineteenth-century mathematical practice, had particular reasons for being concerned with the unity of their discipline. The work on foundations provided mathematics a new thematic unity. Despite Gödel's demonstration of the impossibility of proving the consistency of arithmetic by finite means, the formalist and logicist programs still succeeded in providing methods that enabled mathematicians and philosophers to clarify the logical structure of mathematics. The standard historical accounts that reinforce the idea of the unity of mathematics stemming from the study of foundations were used to legitimate the formal conception of mathematics.

Reevaluating the development of scientific philosophy in the twentieth century requires a full understanding of issues that led to the study of the foundations of mathematics and the development of modern logic. Contextualizing the early work on the foundations of mathematics leads to a different picture of the important role it played in the development of twentieth-century philosophy. The study of the foun-

11. Most of the Couturat-Russell correspondence remains unpublished, but one published reference appears in Russell (1992, 1:210). I thank Kenneth Blackwell and Henrique Carlos Jales Ribeiro for these references.

dations of mathematics is connected to broader philosophical and disciplinary concerns and is not reducible to epistemological concerns within philosophy. Furthermore, the collaborative work of nineteenth-century philosophers and mathematicians who saw philosophy as a science, who used science to answer philosophical questions, belies the widespread assumption that scientific philosophy originated with the logical positivists of the 1920s and 1930s, undermining the logical positivists' claims to priority in making philosophy scientific.

Appendix

Table A1. Occurrence of Topics in Geometry

	Period					
Topic	To 1870	1871–80	1881–90	1891–1900	1901–10	Total
Theory of parallels	470 (78.6)	41 (14)	41 (7.3)	61 (5.5)	79 (5.5)	692 (17)
Foundations of geometry	105 (17.5)	149 (51)	164 (29)	513 (46)	708 (48)	1,639 (41)
No. of dimensions	41 (8)	120 (41)	374 (67)	566 (51)	731 (49)	1,832 (45)

NOTE.—This table shows the number of papers and books written on various topics in geometry and the percentage of the total that they represent, taken from Sommerville's bibliography ([1911] 1970). Theory of parallels is an old topic, and it diminished quickly after 1870. Foundations of geometry peaked around the general acceptance of the non-Euclidean geometries in 1870 and rose again at the turn of the century when interest in the general foundations of mathematics was on the rise. Dimension theory is given for comparison. Numbers in parentheses are percentages.

Table A2. Topics and References in a Random Sample of Standard Undergraduate Geometry Textbooks

Author	Date	History	Logic	Histories Mentioned
Robinson	1940, 1963	Yes	No	None
Wolfe	1945	Yes	No	B, S
Coxteter	1961	Yes	No	B, S
Wylie	1964	Yes	Yes	n.a.
Tuller	1967	Yes	No	B
Golos	1968	Yes	Yes	B
Greenberg	1973	Yes	Yes	B
Hess	1977	Yes	No	n.a.
Kelly and Matthews	1981	Yes	No	n.a.
Martin	1982	Yes	Yes	n.a.
Ryan	1986	Yes	No	M
Trudeau	1987	No	Yes	B, M
Beneditti and Petronio	1991	No	No	M
Perry	1992	No	Yes	n.a.
Stahl	1993	Yes	No	B, M

NOTE.—All of these texts treat geometry axiomatically and all mention some historical figures. I have marked those that emphasize history by having a separate section, those that emphasize axiomatics and logic by introducing it generally rather than only in a geometric context, and also noted the histories of non-Euclidean geometry mentioned in the bibliography (B = Bonola [1912]; S = Stäckel and Engel [1895]; M = Milnor [1982]; n.a. = no bibliography). From 1940 to 1986, all of the texts in this survey have a separate historical section; of those that have a bibliography, only two do not list Bonola. From 1987 on, only one of the texts has a separate historical section, and only one lists Bonola. However, from 1982 on, all those with bibliographies list Milnor (1982).

References

Albers, Donald J., Gerald L. Alexanderson, and Constance Reid. 1986. *International Mathematical Congresses: An Illustrated History, 1893–1986*. New York: Springer.

Bayle, Pierre, Pierre Desmaizeaux, Alexis Gaudin, and Anthelme de Tricaud. 1695–1702. *The Dictionary Historical and Critical of Mr. Peter Bayle*. London: J. J. & P. Knapton.

Beaulieu, Liliane. 1994. "Dispelling a Myth: Questions and Answers about Bourbaki's Early Works, 1934–1944." Pp. 241–52 in *The Intersection of History and Mathematics*. Edited by Chikara Sasaki, Mitsuo Sigiura, and Joseph Warren Dauben. Basel and Boston: Birkhauser.

Benacerraf, Paul. 1981. "Frege: The Last Logicist." Pp. 41–67 in *Frege's*

Philosophy of Mathematics. Edited by William Demopoulos. Cambridge, Mass.: Harvard University Press.

Benedetti, R., and Carlo Petronio. 1992. *Lectures on Hyperbolic Geometry*. Berlin: Springer.

Bishop, Errett. 1975. "The Crisis in Contemporary Mathematics." *Historia Mathematica* 2 (November): 507–17.

Boethius. 1962. *The Consolation of Philosophy*. Translated by Green Richard. New York: MacMillan.

Bonola, Roberto. 1906. *La Geometria non-Euclidea: Esposizione Storico-Critica del suo Sviluppo*. Bologna: Nicola Zanichelli.

———. (1912) 1955. *Non-Euclidean Geometry: A Critical and Historical Study of its Development*. Translated by H. S. Carslaw. New York: Dover.

Boole, George. (1854) 1958. *An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities*. London; reprint, New York: Dover.

Boolos, George. 1994. "The Advantages of Honest Toil over Theft." Pp. 27–44 in *Mathematics and Mind*. Edited by Alexander George. New York: Oxford University Press.

Bourbaki, Nicolas. 1948. "L'architecture des mathématique." Pp. 35–47 in *Les Grands Courants de la Pensée Mathématique*. Edited by François Le Lionnais. Paris: Cahiers du Sud.

———. 1971. "The Architecture of Mathematics" Pp. 23–36 in *Great Currents of Mathematical Thought*. Vol. 1, *Mathematics: Concepts and Development*. Edited by F. Le Lionnais. New York: Dover.

Bowne, G. D. 1966. *The Philosophy of Logic*. The Hague: Mouton.

Bynum, Terrell Ward. 1993. "The Evolution of Frege's Logicism." Pp. 193–200 in *Logic and Foundations of Mathematics in Frege's Philosophy*. Edited by Hans D. Sluga. New York: Garland.

Cajori, Florian. 1919. *A History of Mathematics*. 2d ed. New York: MacMillan, 1938.

Carruccio, Ettore. (1951) 1964. *Mathematics and Logic in History and in Contemporary Thought*. London: Faber and Faber.

Cayley, Arthur. 1883. "Presidential Address to the British Association at Southport." *Report of the British Association for the Advancement of Science (Southport)* 53:3–37.

Coffa, J. Alberto. 1986. "From Geometry to Tolerance: Sources of Conventionalism in Nineteenth-Century Geometry." Pp. 3–70 in *From Quarks to Quasars: Philosophical Problems of Modern Physics*. Edited by Robert G. Colodny. Pittsburgh: University of Pittsburgh Press.

———. 1991. *The Semantic Tradition from Kant to Carnap To the Vienna*

Station. Edited by Linda Wessels. Cambridge: Cambridge University Press.

Contro, W. S. 1976. "Von Pasch zu Hilbert." *Archive for the History of the Exact Sciences* 15:285–95.

Coolidge, Julian Lowell. 1909. *The Elements of Non-Euclidean Geometry.* London: Oxford University Press.

______. (1940) 1963. *A History of Geometrical Methods.* New York: Dover.

Couturat, Louis. (1901) 1969. *La Logique de Leibniz d'après des Documents Inédits.* Hildesheim: Georg Olms.

______. 1904. "Logique et Philosophie des Science: Séances de Section et Séances Générales." *Revue de Métaphysique et de Morale* 12:1037–77.

______. 1905. *Les Principes des Mathématiques, avec un Appendice sur la Philosophie des Mathématiques de Kant.* Paris: F. Alcan.

Couturat, Louis, ed. 1903. *Opuscules et Fragments Inédits de Leibniz: Extraits des Manuscrits de la Bibliothéque Royale de Hanovre.* Paris: Felix Alcan.

Coxeter, H. S. M. 1969. *Introduction to Geometry.* 2d ed. New York: Wiley.

Crawford, Elisabeth T. 1992. *Nationalism and Internationalism in Science, 1880–1939: Four Studies of the Nobel Population.* Cambridge: Cambridge University Press.

Curry, Haskell B. 1963. *Foundations of Mathematical Logic.* New York: McGraw-Hill.

Demopoulos, William. 1981. "Frege and the Rigorization of Analysis." Pp. 68–88 in *Frege's Philosophy of Mathematics.* Edited by William Demopoulos. Cambridge, Mass.: Harvard University Press.

Detlefsen, Michael. 1994. "Constructivism." Pp. 656–64 in *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences.* Edited by Ivor Grattan-Guinness. London: Routledge.

Eves, Howard Whitley. 1983. *An Introduction to the History of Mathematics.* 5th ed. Philadelphia: Saunders.

Frege, Gottlob. (1879) 1972. *Conceptual Notation, and Related Articles.* Translated by Terrell Ward Bynum. Oxford: Clarendon Press.

______. (1880/81) 1979. "Boole's Logical Calculus and the Conceptscript." Pp. 9–46 in *Posthumous Writings.* Edited by Hans Hermes, Friedrich Kambartel, and Friedrich Kaulbach. Chicago: University of Chicago Press.

______. (1884) 1953. *The Foundations of Arithmetic.* Translated by J. L. Austin. Oxford: Blackwell.

______. (1914) 1979. "Logic in Mathematics." Pp. 203–50 in *Posthumous*

Writings. Edited by Hans Hermes, Friedrich Kambartel, and Friedrich Kaulbach. Chicago: University of Chicago Press.

Freudenthal, Hans. 1957. "Zur Geschichte der Grundlagen der Geometrie." *Nieuw Archief voor Wiskunde* 5:105–42.

———. 1974. "The Impact of Von Staudt's Foundations of Geometry." Pp. 189–99 in *For Dirk Struik: Scientific, Historical, and Political Essays Honor of Dirk Struik*. Edited by R. S. Cohen, J. J. Stachel, and M. W. Wartofsky. Dordrecht: Reidel.

Galdeano, Zoel. 1897. "L'unification des concepts dans les mathématiques." *Verhandlungen des Internationale Mathematiker-Kongresses* 1: 227–231.

Galison, Peter, and David J. Stump, eds. 1996. *The Disunity of Science: Boundaries, Contexts, and Power.* Stanford, Calif.: Stanford University Press.

Garciadiego, Alejandro Ricardo. 1992. *Bertrand Russell and the Origins of the Set-Theoretic 'Paradoxes'*. Basel and Boston: Birkhauser.

———. 1994. "The Set-Theoretic Paradoxes." Pp. 629–34 in *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences.* Edited by Ivor Grattan-Guinness. London: Routledge.

Giuculescu, Alexandre. 1983. "Un Promoteur des Recheches Interdisciplinaires: Louis Couturat (1868–1914)." Pp. 113–24 in *L'Oeuvre de Louis Couturat (1868–1914): De Leibniz à Russel.* Paris: Presses de l'École Normale Supérieure.

Golos, Ellery B. 1968. *Foundations of Euclidean and Non-Euclidean Geometry.* New York: Holt, Rinehart & Winston.

Graham, Loren, Wolf Lepenies, and Peter Weingart, eds. 1983. *Functions and Uses of Disciplinary History.* Boston: D. Reidel.

Grattan-Guinness, Ivor. 1984. "From Weierstrass to Russell: A Peano Medley." *Rivista di Storia della Scienza* 5, no. 2, pp. 1–16.

———. 1985. "Bertrand Russell's Logical Manuscripts: An Apprehensive Brief." *History and Philosophy of Logic* 6:53–74.

———. 1986. "Russell's Logicism versus Oxbridge Logics, 1890–1925." *Russell* 5:101–131.

———. 1988. "Living Together and Living Apart: On the Interactions between Mathematics and Logics from the French Revolution to the First World War." *South African Journal of Philosophy* 7 no. 2, pp. 73–82.

Gray, Jeremy. (1979) 1989. *Ideas of Space: Euclidean, Non-Euclidean and Relativistic.* Oxford: Clarendon Press.

———. 1986. *Linear Differential Equations and Group Theory from Riemann to Poincaré.* Boston: Birkhauser.

———. 1987. "The Discovery of Non-Euclidean Geometry." Pp. 37–60 in *Studies in the History of Mathematics*. Edited by Esther R. Phillips. Washington, D.C.: Mathematical Association of America.

———. 1992. "The Nineteenth-Century Revolution in Mathematical Ontology." Pp. 226–48 in *Revolutions in Mathematics*. Edited by Donald Gillies. Oxford: Oxford University Press.

———. 1994. "Projective Geometry." Pp. 897–907 in *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*. Edited by Ivor Grattan-Guinness. London: Routledge.

Greenberg, Marvin Jay. 1980. *Euclidean and Non-Euclidean Geometries: Developments and History*. 2d ed. San Francisco: Freeman.

Griffin, Nicholas. 1991. *Russell's Idealist Apprenticeship*. Oxford: Oxford University Press.

Hager, Paul J. 1994. *Continuity and Change in the Development of Russell's Philosophy*. Dordrecht: Kluwer Academic.

Hallett, Michael. 1994. "Hilbert's Axiomatic Method and the Laws of Thought." Pp. 158–200 in *Mathematics and Mind*. Edited by Alexander George. New York: Oxford University Press.

Hess, Adrien. 1977. *Four-Dimensional Geometry: An Introduction*. Reston, Va.: National Council of Teachers of Mathematics.

Hilbert, David. (1901) 1932–35. "Mathematische Probleme." Pp. 290–329 in *Gesammelte Abhandlungen*. Berlin: Springer.

———. 1902. "Mathematical Problems." *Bulletin of the American Mathematical Society* 8:437–79.

———. (1904) 1967. "On the Foundations of Logic and Arithmetic." Pp. 129–38 in *From Frege to Godel; a Source Book in Mathematical Logic, 1879–1931*. Edited by Jean Van Heijenoort. Cambridge, Mass.: Harvard University Press.

———. (1917) 1932–35. "Axiomatisches Denken." Pp. 146–56 in *Gesammelte Abhandlungen*. Berlin: Springer.

———. 1921. *The Foundations of Geometry*. Translated by E. J. Townsend. Chicago: Open Court.

Hoüel, Jules. 1867. *Essai Critique sur les Principes Fondamentaux de la Géométrie Élémentaire ou Commentaire sur les XXXII Premiéres Propositions des Éléments d'euclide*. Paris: Gauthier-Villars.

Hylton, Peter. 1990. *Russell, Idealism, and the Emergence of Analytic Philosophy*. Oxford: Clarendon Press.

Johnson, Dale M. 1981. "The Problem of the Invariance of Dimension in the Growth of Modern Topology. Part II" *Archive for the History of Exact Sciences* 25:85–267.

Kárteszi, Ferenc, ed. 1987. János Bolyai, *Appendix: The Theory of Space*. Amsterdam: North-Holland.

Kelly, Paul, and Gordon Matthews. 1981. *The Non-Euclidean, Hyperbolic Plane Its Structure and Consistency.* New York: Springer.

Kennedy, Hubert C. 1980. *Peano: Life and Works of Giuseppe Peano.* Dordrecht: D. Reidel.

Klein, Felix. (1893) 1896. "The Present State of Mathematics." Pp. 133–35 in *Mathematical Papers Read at the International Mathematical Congress, Chicago, 1893.* Edited by E. Hastings Moore. New York: MacMillan.

Kline, Morris. 1972. *Mathematical Thought from Ancient to Modern Times.* New York: Oxford University Press.

———. 1980. *Mathematics: The Loss of Certainty.* New York: Oxford University Press.

Krentz, Edgar. 1975. *The Historical-Critical Method.* Philadelphia: Fortress.

Laita, L. M. 1980. "Boolean Algebra and its Extra-logical Sources: The Testimony of Mary Everest Boole." *History and Philosophy of Logic* 1:37–60.

Laudan, Rachel. 1993. "Histories of Science and Their Uses: A Review to 1913." *History of Science* 31:1–34.

Lie, Sophus. 1895. *Untersuchungen über unendliche continuirliche Gruppen.* Leipzig: S. Hirzel.

Lyons, F. S. L. 1963. *Internationalism in Europe, 1815–1914.* Leyden: A. W. Sythoff.

Mach, Ernst. 1891. *The Science of Mechanics: A Critical and Historical Exposition of its Principles.* Chicago: Open Court.

Mac Lane, Saunders. 1988. "Concepts and Categories in Perspective." Pp. 323–63 in *A Century of Mathematics in America,* Part 1. Edited by Peter Duren. Providence, R.I.: American Mathematical Society.

Manning, Henry P. 1901. *Non-Euclidean Geometry.* Boston: Ginn.

Martin, George E. 1982. *The Foundations of Geometry and the Non-Euclidean Plane.* New York: Springer.

Mathews, G. B. 1898. "Review of Whitehead, 1898, *A Treatise on Universal Algebra.*" *Nature* 58:385–86.

May, Kenneth O. 1975. "Historical Vices I. Logical Attribution." *Historia Mathematica* 2:185–87.

Meschkowski, Herbert. 1965. *Non-Euclidean Geometry.* Translated by Abe Shenitzer. New York: Academic Press.

Meyer, François, ed. 1899–1916. *Encyklopädie der Mathematischen Wissenschaft mit Einschluss ihre Anwendungen.* Leipzig: B. G. Teubner.

Milnor, John. 1982. "Hyperbolic Geometry: The First 150 years." *American Mathematical Society Bulletin* 6:9–24.

Molk, Jules, ed. 1904–1916. *Encyclopédie des Sciences Mathématiques Pures et Appliquées.* Paris: Gauthier Villars.

Moore, Gregory H. 1978. "The Origins of Zermelo's Axiomatization of Set Theory." *Journal of Philosophical Logic* 7:307–29.

Moritz, Robert Edouard. (1914) 1958. *Memorabilia Mathematica; or, The Philomath's Quotation-Book.* New York: Dover.

Nagel, Ernest. (1939) 1979. "The Formation of Modern Conceptions of Formal Logic in the Development of Geometry." *Osiris* 7:142–224. Reprinted pp. 195–259 in *Teleology Revisited and Other Essays in the Philosophy and History of Science.* New York: Columbia University Press.

Nowak, Gregory. 1989. "Riemann's *Habilitationsvortrag* and the Synthetic *A Priori* Status of Geometry." Pp. 17–46 in *The History of Modern Mathematics.* Vol. 1, *Ideas and Their Reception.* Edited by David E. Rowe and John McCleary. Boston: Academic.

Paul, Harry W. 1968. "The Debate over the Bankruptcy of Science in 1895." *French Historical Studies* 5:299–327.

Peano, Giuseppe. 1891. "Principii di logica matematica." *Rivista di Matematica* 1:1–10.

———. 1894*a*. *Notations de Logique Mathematique: Introduction au Formulaire de Mathematique Publie par la "Rivista di matematica."* Turin: C. Guadagnini.

———. 1894*b*. "Sui fondamenti della Geometria." *Rivista di Matematica* 4:51–90.

———. 1901. *Formulaire Mathématique.* Vol. 3. Paris: Gauthier Villars.

———. 1973. *Selected Works of Giuseppe Peano.* Edited and translated by Hubert C. Kennedy. Toronto: University of Toronto Press.

Peckhaus, Volker. 1994. "Hilbert's Axiomatic Programme and Philosophy." Pp. 91–112 in *The History of Modern Mathematics.* Vol. 3, *Images, Ideas, and Communities.* Edited by Eberhard Knobloch and David E. Rowe. Boston: Academic.

Perry, Earl. 1992. *Geometry: Axiomatic Developments with Problem Solving.* New York: M. Dekker.

Pierpont, James P. 1905. "The History of Mathematics in the Nineteenth Century." Pp. 474–493 in *Congress of Arts and Science: Universal Exposition, St. Louis, 1904.* Edited by Howard J. Rogers and Hugo Münsterberg. New York: Houghton, Mifflin.

Poincaré, Henri. 1904. *Rapport sur les Travaux de M. Hilbert, professeur à l'Université de Goettingen, Présentés au troisième concours du prix Lobatchefsky.* Kasan: Imperial University Press.

Provisional International Committee. (1901) 1968. *International Cat-*

elogue of Scientific Literature: Part A Mathematics. New York: Royal Society.

Pyenson, Lewis. 1983. *Neohumanism and the Persistence of Pure Mathematics in Wilhelmian Germany.* Philadelphia: American Philosophical Society.

Reid, Constancel. 1970. *Hilbert.* Berlin: Springer.

Rempel, Richard. 1979. "From Imperialism to Free Trade: Couturat, Halevy, and Russell's First Crusade." *Journal of the History of Ideas* 40:423–43.

Répertoire Bibliographique des Sciences Mathématiques. 1893. *Index du Répertoire Bibliographique des Sciences Mathématiques.* Paris: Gauthier-Villars.

Robinson, Gilbert de Beauregard. (1940) 1963. *The Foundations of Geometry.* Toronto: University of Toronto Press.

Rodríguez-Consuegra, F. A. 1991. *The Mathematical Philosophy of Bertrand Russell: Origins and Development.* Boston: Birkhauser.

Rosenfeld, B. A. 1988. *A History of Non-Euclidean Geometry: Evolution of the Concept of a Geometric Space.* Translated by Abe Shenitzer. New York: Springer.

Russell, Bertrand. 1900. *A Critical Exposition of the Philosophy of Leibniz, with an Appendix of Leading Passages.* London: G. Allen & Unwin.

______. (1901) 1990 "Recent Italian Work on the Foundations of Mathematics." Pp. 350–62 in *Towards the "Principles of Mathematics" 1900–02.* Edited by Gregory H. Moore. Vol. 3 in *The Collected Papers of Bertrand Russell.* London: Routledge.

______. 1903. *Principles of Mathematics.* Cambridge; reprint, New York: Norton.

______. 1985. *Contemplation and Action, 1902–14.* Edited by Richard A. Rempel, Andrew Brink, and Margaret Moran. Vol. 12 in *The Collected Papers of Bertrand Russell.* London: Routledge.

______. 1992. *The Selected Letters of Bertrand Russell.* Boston: Houghton Mifflin.

Ryan, Patrick J. 1986. *Euclidean and Non-Euclidean Geometry: An Analytical Approach.* Cambridge: Cambridge University Press.

Sanzo, Ubaldo. 1983. "Philosophie et Science dans la Pensée de Louis Couturat." Pp. 69–80 in *L'Oeuvre de Louis Couturat (1868–1914): De Leibniz à Russell.* Edited by École Normale Supérieure. Paris: Presses de l'École Normale Supérieure.

Schroeder-Gudehus, Brigitte. 1990. "Nationalism and Internationalism." Pp. 909–19 in *Companion to the History of Modern Science.* Edited by Robert C. Olby. London: Routledge.

Sluga, Hans D. 1980. *Gottlob Frege.* London: Routledge & Kegan Paul.

Smith, G. C. 1983. "Boole's Annotations on 'The Mathematical Analysis of Logic.'" *History and Philosophy of Logic* 4:27–39.

Sommerville, D. M. Y. (1911) 1970. *Bibliography of Non-Euclidean Geometry.* London; reprint, New York: Chelsea.

Stäckel, Paul, and Friedrich Engel. 1895. *Die Theorie der Parallellinien von Euklid bis auf Gauss.* Leipzig: Teubner.

Stahl, Saul. 1993. *The Poincaré Half-Plane: a Gateway to Modern Geometry.* Boston: Jones & Bartlett.

Steiner, H. G. 1964. "Frege und die Grundlagen der Geometrie II." *Mathematische-Physikalische Semesterberichte,* new series, 11:35–47.

Stump, David J. 1996. "Poincaré's Curious Role in the Formalization of Mathematics." Pp. 481–92 in *Henri Poincaré: Science and Philosophy, International Congress Nancy, France, 1994.* Edited by Jean Louis Greffe, Gerhard Heinzmann, and Kuno Lorenz. Paris and Berlin: A. Blanchard & Akademie.

Trudeau, Richard J. 1987. *The Non-Euclidean Revolution.* Boston: Birkhauser.

Tuller, Annita. 1967. *A Modern Introduction to Geometries.* Toronto: Van Nostrand.

Whitehead, Alfred North. (1898) 1960. *A Treatise on Universal Algebra, with Applications.* Cambridge; reprint, New York: Hafner.

———. (1911) 1958. *An Introduction to Mathematics.* London; reprint, New York: Oxford University Press.

Whitehead, Alfred North, and Bertrand Russell. (1910) 1913. *Principia Mathematica.* Cambridge: Cambridge University Press.

Wilder, Raymond L. 1952. *The Foundations of Mathematics.* New York: Wiley.

Wolfe, Harold E. 1945. *Introduction to Non-Euclidean Geometry.* New York: Holt, Rinehart & Winston.

Wylie, Clarence Raymond. 1964. *Foundations of Geometry.* New York: McGraw-Hill.

Young, J. W. A., and Lambert L. Jackson. 1911. *A First Course in Elementary Algebra.* New York: D. Appleton.